ON THE DISTRIBUTION THEORY FOR SOME CONSTRAINED LIFE TESTING EXPERIMENTS

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THESIS

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ABSTRACT

Various aspects of reliability test programs when the number of items to be tested equals or exceeds the number of test chambers available for testing and when the underlying failure distribution is exponential are studied. Specifically, two bivariate processes are formed and characterized; (i) the time until the r^{th} failure and the accumulated test time of the first r failures and (ii) the number of failures in the time interval (0,T) and the total test time accumulated in (0,T). These are treated in the two cases; replacement and non-replacement. Relations which exist between contractual specifications for reliability demonstration and the underlying distributions of the test program are discussed. Maximum likelihood estimators are developed for the distributions when necessary.

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TABLE OF SYMBOLS AND ABBREVIATIONS

i	Number of items available for testing
c	Number of test chambers available for testing
p	$i-c$
T_j	The time of the j^{th} failure measured from test start
B_j	The total test time accumulated up to the j^{th} failure
B_{Υ}	The total test time accumulated up to time Υ
S_j	$T_j - T_{j-1}$
N_t	Number of items to have failed in time t
$f^*(s)$	Laplace Transform of $f(t)$

I. INTRODUCTION

A reliability program manager shall be considered as one who has the task to demonstrate, according to contractual specifications, the reliability of a piece of equipment. Such a manager is commonly faced with certain physical constraints such as limited test facilities, number of items available for testing, time available to complete the test program, etc. The manager is also faced with various costing considerations such as the cost of the items to be tested, or keeping the limited test facilities in use for a specific equipment vice another, etc. In view of the preceeding, the manager is concerned with such problems as, "what is the probability of successful demonstration of the specified reliability throughout the test program," "what is an estimation of the time required to complete the test program," and "what is the number of items which will fail during the testing." It is the purpose of this paper to provide mathematical answers to questions of this type and hence be an aid to such a manager.

Throughout this paper it will be assumed that all items to be tested have an exponential failure density (i.e. $f(x) = \lambda e^{-\lambda x}$ and λ is the failure rate of the item to be tested.) The assumption of exponential failure density is often made for electronic devices and it has been shown [Ref. 4] that complex equipments tend to have an exponential failure density. This assumption also leads to a large number of mathematical simplifications. The extension to other failure laws is difficult and the present analysis can serve as a skeleton for that extension. The mathematical developments in this paper heavily rely upon the properties of the Laplace Transform.

It is a common practice to write contractual specifications for reliability demonstration which call for the demonstration of a specified lower confidence limit on reliability, R_l, with a confidence coefficient γ where the item is required to operate over a mission time T. Lloyd and Lipow [Ref. 4] show that this specification gives rise to the demonstration of an upper confidence limit λ_0 for λ with confidence coefficient γ when the failure density is exponential. This means that the estimator $\hat{\lambda}$ for λ, the failure rate, must be less than λ_0 in order to assert that the specified reliability has been demonstrated.

Truncated life testing [Ref. 4] is a technique of experimental design that economizes in terms of either the number of items tested or the amount of time spent testing, or both. This paper is concerned with those designs that terminate after either a

(1) fixed number of failures or

(2) fixed amount of time.

Other (sequential) stopping rules are popular [Ref. 4], but will not be considered. The design of the experiment will trade cost against the utility of success. The total cost is the sum of the number of items tested multiplied by the cost per item and the amount of time spent testing multiplied by the cost per unit time (for use of the test facilities.) Thus the total cost is a random number because the time spent testing for stopping rule (1) is a random variable and the number of items tested is random for stopping rule (2). Thus the total expected cost is a function of λ and the number of failures for stopping rule (1), and is a function of λ and the termination time for stopping rule (2). Generally the expected cost will increase as the failure rate λ decreases. On the other hand, the lower the failure rate the more desirable is the equipment and the more likely it is that the contractual specification ($\hat{\lambda} \leq \lambda_0$) can be demonstrated. It follows that the joint distribution of (1) the estimator and the terminal time and (2) the estimator and the number of failures will play a key role in balancing cost of testing against the value of successful completion. This latter value is a utility type consideration and must be developed separately for each particular kind of item. A similar statement applies to the cost coefficients. It should be noted that many modern systems have high item cost and high test facility usage costs. Only the methodology for treating the joint distributions mentioned above can be treated in general. Also, the program manager has the option of terminating a test program early if the chances of successful completion diminish during the course of testing. Such a decision should be based on probability statements concerning the aforementioned random vectors conditioned by the current status of affairs. Thus an understanding of these quantities is needed when they are viewed as bivariate stochastic processes.

It is assumed that the program manager is constrained by a limited number, c, of test chambers. If the number of items available for testing, i, is less than c, then all items are put on test at the same time and testing takes place without replacement. Without loss of generality this case can be indexed by $i = c$. Otherwise $i > c$ and the procedure is to test with replacement until the reservoir of spares is depleted and then continue to test without replacement. The case $i = c$ is treated in Chapter II and the case $i > c$ is treated in Chapter III. The joint distributions and the marginal distributions and the moments for each type stopping rule are developed. It has been shown by Read [Ref. 5] that the normal approximations to these distributions are valid if the number of items failed and the failure rate are not too small. Explicit distributions will be necessary when these quantities are small. An indication for the necessary size of these quantities for the normal approximations to be valid is also given by Read [Ref. 5].

II. THE NUMBER OF ITEMS TO BE TESTED DOES NOT EXCEED THE NUMBER OF TEST CHAMBERS

A. STOPPING RULE: STOP AT r FAILURES

In this first case ($i=c$ or $p=0$), all items are put on test at the start of the testing program and the test is complete when r of the c items have failed.

Let T_1 = Time to first failure from $t=0$

T_2 = Time to second failure from $t=0$

.

.

T_r = Time to r^{th} failure from $t=0$

and $S_1 = T_1$

$S_2 = T_2 - T_1$

.

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$S_r = T_r - T_{r-1}$

then S_j has density $(c-j+1)\lambda e^{-(c-j+1)\lambda x}$, $x > 0$. The S_j are independent and hence the Laplace Transform of the joint density of the S_j's is:

$$f^*(s_1, s_2, \ldots, s_r) = \prod_{j=1}^{r} \frac{(c-j+1)\lambda}{s_j + (c-j+1)\lambda} \qquad (1)$$

It is of general interest to characterize the distribution of the order statistic. Let the Laplace Transform of the joint density of $T_1, T_2, \ldots, T_r$ be:

$$f^*(u_1, u_2, \cdots u_3) = E\left[e^{-\sum_{j=1}^{r} u_j T_j} \right]$$

but $T_j = \sum_{i=1}^{j} S_i$, and hence,

$$f^*(u_1, u_2, \cdots u_r) = E\left[e^{-\sum_{j=1}^{r} u_j \left[\sum_{i=1}^{j} S_i\right]} \right]$$

$$= E\left[e^{-\sum_{i=1}^{r} S_i \left[\sum_{j=i}^{r} u_j\right]} \right]$$

Therefore, letting $\Delta_j = \sum_{k=j}^{r} u_k$ yields:

$$f^*(u_1, u_2, \cdots u_r) = \prod_{j=1}^{r} \frac{(c-j+1)\lambda}{(c-j+1)\lambda + \sum_{k=j}^{r} u_k} \tag{2}$$

Inverting equation (2) on u_1 gives:

$$f^*(\tau_1, u_2, \cdots, u_r) = \frac{c(c-1)\lambda^2 e^{-\left\{(c-1)\lambda + u_2 + \sum_{k=3}^{r} u_k\right\}\tau_1}}{e^{\lambda\tau_1}\left\{(c-1)\lambda + u_2 + \sum_{k=3}^{r} u_k\right\}} \cdot$$

$$\prod_{j=3}^{r} \frac{(c-j+1)\lambda}{(c-j+1)\lambda + \sum_{k=j}^{r} u_k}$$

and noting that the Laplace Transform of

$$e^{-\alpha x}\, U(x-\tau_1)$$

where $U(x-t_1)$ is the unit step function translated to t_1, is

$$\frac{e^{-(s+\alpha)\tau_1}}{(s+\alpha)}$$

gives rise to the joint density of $T_1, T_2, \ldots, T_r$ as:

$$f(\tau_1, \tau_2, \ldots, \tau_r) = \frac{c!\,\lambda^r}{(c-r)!}\, e^{-\lambda \sum_{j=1}^{r-1} \tau_j}\, e^{-\lambda(c-r+1)\tau_r} \qquad (3)$$

for $0 < t_1 < t_2 < \cdots < t_r$

The marginal distribution of the time to the j^{th} failure can be calculated easily from equation (2).

It has been shown [Ref.2] that the maximum likelihood estimator for λ upon completion of the test program is:

$$\hat{\lambda} = \frac{r}{B_r} \qquad (4)$$

where B_r is the total accumulated test time of all items. (Alternatively, this may be shown from equation (3)). Thus the joint distribution of $\hat{\lambda}$ and the time to completion T_r is tantamount to the joint distribution of T_r and B_r. This joint distribution has been characterized by Read [Ref.5] and for completeness the Laplace Transform and the density is recorded:

$$f^*(u_1,u_2) = \prod_{j=1}^{r} \frac{(c-j+1)\lambda}{u_1 + (\lambda+u_2)(c-j+1)} \tag{5}$$

$$f_{T_r,B_r}(x_1,x_2) = r\lambda^r \sum_{j=0}^{r} (-1)^{r-j-1} \binom{r-1}{j} e^{-x_2\lambda} \frac{[x_2 - x_1(c-j)]^{r-2}}{(r-2)!} \cdot U[x_2 - x_1(c-j)] \tag{6}$$

where U(x) is the unit step function.

When the program manager desires to mark progress at some time τ during the course of testing, the data available at that time will be a complete history of the testing including N_τ and B_τ , i.e. the current tally on the number of failures and the accumulated total test time. Due to the fact that the exponential distribution has no memory, all data other than N_τ and B_τ are irrelevant in terms of making updated probability statements concerning T_r and B_r. It also follows that the conditional density of T_r, B_r given that N_τ =m (for m < r) and B_τ=b can be characterized from equations (5) and (6) by replacing r with (r−m), c with (c−m) and by translating T_r to $T_{r-m} + \tau$ and B_r to B_{r-m} + b. Thus:

$$f^*_{T_r,B_r|N_\tau=m,B_\tau=b}(u_1,u_2) = e^{-u_1\tau-u_2 b} \prod_{j=1}^{r-m} \frac{(c-m-j+1)\lambda}{u_1 + (u_2+\lambda)(c-m-j+1)} \tag{7}$$

and letting k = r−m yields:

$$f_{T_r, B_r \mid N_\tau = m,\, B_\tau = b}(x_1, x_2) =$$

$$k \lambda^k \binom{c-m}{k} \sum_{j=0}^{k-1} (-1)^{k-j-1} \binom{k-1}{j} e^{-(x_2 - b)} \cdot \tag{8}$$

$$\frac{\left[(x_2 - b) - (x_1 - \tau)(c - m - j) \right]^{k-2}}{(k-2)!} \; U\left\{ (x_2 - b) - (x_1 - \tau)(c - m - j) \right\}$$

The univariate distributions developed in the same manner from equation (5) are:

$$f_{T_r}(x_1) = \binom{c}{r} r \lambda e^{-\lambda x_1 (c - r + 1)} \left[1 - e^{-\lambda x_1} \right]^{r-1}, \quad x_1 > 0 \tag{9}$$

$$f_{T_r \mid N_\tau = m,\, B_\tau = b}(x_1) =$$

$$\binom{c-m}{k} k \lambda e^{-\lambda (x_1 - \tau)(c - r + 1)} \left[1 - e^{-\lambda (x_1 - \tau)} \right]^{k-1}, \quad x_1 > \tau \tag{10}$$

$$f_{B_r}(x_2) = \frac{\lambda (\lambda x_2)^{r-1} e^{-\lambda x_2}}{(r-1)!}, \quad x_2 > 0 \tag{11}$$

$$f_{B_r \mid N_\tau = m,\, B_\tau = b}(x_2) =$$

$$\frac{\lambda (\lambda [x_2 - b])^{k-1} e^{-\lambda (x_2 - b)}}{(k-1)!}, \quad x_2 > b \tag{12}$$

The moments needed for mean value analysis and for the normal approximations developed by Read [Ref.5] are:

$$E\{T_r\} = \frac{1}{\lambda} \sum_{j=1}^{r} \frac{1}{(c-j+1)} \tag{13}$$

$$VAR\{T_r\} = \frac{1}{\lambda^2} \sum_{j=1}^{r} \frac{1}{(c-j+1)^2}$$

$$E\{T_r \mid N_\tau = m, B_\tau = T\} = \tau + \frac{1}{\lambda} \sum_{j=m+1}^{r} \frac{1}{(c-j+1)}$$

$$VAR\{T_r \mid N_\tau = m, B_\tau = T\} = \frac{1}{\lambda^2} \sum_{j=m+1}^{r} \frac{1}{(c-j+1)^2}$$

$$E\{B_r\} = \frac{r}{\lambda}$$

$$VAR\{B_r\} = \frac{r}{\lambda^2}$$

$$E\{B_r \mid N_\tau = m, B_\tau = T\} = T + \frac{r-m}{\lambda}$$

$$VAR\{B_r \mid N_\tau = m, B_\tau = T\} = \frac{r-m}{\lambda^2}$$

$$COV\{T_r, B_r\} = \frac{1}{\lambda^2} \sum_{j=0}^{r-1} \frac{1}{(c-j)}$$

$$COV\{T_r, B_r \mid N_\tau = m, B_\tau = T\} = \frac{1}{\lambda^2} \sum_{j=0}^{r-m-1} \frac{1}{(c-m-j)}$$

Now consider those problems posed for the reliability program manager. The manager can successfully demonstrate reliability if he can produce an estimator for λ, say $\hat{\lambda}$, such that $\hat{\lambda} \leq \lambda_0$ where λ_0 is an upper confidence limit for λ with confidence coefficient γ. Now upon completion of the test program, the manager will have successfully demonstrated reliability if:

$$\hat{\lambda} = \frac{r}{B_r} \leq \lambda_0 \qquad (14)$$

Therefore, it is required that the total test time accumulated during the test program B_r must be greater than r/λ_0 where B_r has density given by equation (11). Hence, if the program manager is willing to use a design estimate, say λ_D, for λ, he has available to him, the probability of successfully demonstrating reliability, an estimate of the total time for the test program, and various other pieces of information prior to the start of the testing.

Now given that at some time, τ, prior to completion of the test, the manager desires an estimator for λ based upon the data which is available at τ, it has been shown [Ref. 2], that the maximum likelihood estimator $\hat{\lambda}_\tau$ is

$$\hat{\lambda}_\tau = \frac{N_\tau}{B_\tau}$$

and by simply upgrading the distributions in question with $\hat{\lambda}_\tau$, the program manager has a continuing estimate of the most important characteristics of his test program.

B. STOPPING RULE: STOP AT A FIXED TIME T

The distribution of the order statistic $T_1, T_2, \ldots T_r$ given $N_T = r$ has general interest and can be obtained from equation (3) as follows:

$$f_{T_1,T_2,\ldots,T_r \mid N_T = r}(\tau_1,\tau_2,\ldots,\tau_r) = \frac{\text{Prob}\left\{N_T = r \mid T_r = \tau_r\right\} f_{T_1,T_2,\ldots,T_r}(\tau_1,\tau_2,\ldots,\tau_r)}{\text{Prob}\left\{N_T = r\right\}}$$

Noting:

$$\text{Prob}\left\{N_T = r \mid T_r = \tau_r\right\} = e^{-(c-r)\lambda(T-\tau_r)}, \quad \tau_r \doteq T$$

$$\text{Prob}\left\{N_T = r\right\} = \binom{c}{r}\left[1 - e^{-\lambda T}\right]^r e^{-\lambda T(c-r)}$$

and therefore

$$f_{T_1,T_2,\ldots,T_r \mid N_T = r}(\tau_1,\tau_2,\ldots,\tau_r) = \frac{r!\,\lambda^r e^{-\lambda \sum_{i=1}^{r} \tau_i}}{\left[1 - e^{-\lambda T}\right]^r} \tag{15}$$

$$0 < t_1 < t_2 < \ldots < t_r < T$$

Under a fixed time stopping rule, with i = c, it has been shown [Ref.2] that the maximum likelihood estimator for λ upon completion of the test program is:

$$\hat{\lambda} = \frac{N_T}{B_T} \tag{16}$$

and the joint distribution of N_T, B_T is needed for experimental design and decision making. Recognizing that N_T is a discrete random variable while B_T is a continuous random variable, their joint density will be referred to as:

$$f_{N_T, B_T}(r, b) = \frac{\partial \left(Prob\left(N_T = r,\ B_T \leq b \right) \right)}{\partial b} \tag{17}$$

To develop this quantity, the following relationship is needed:

$$B_T = B_{N_T} + (T - T_{N_T})(C - N_T)$$

and hence:

$$P\left(N_T = r, B_T \leq b \right) =$$

$$P\left(T_r \leq T,\ T_{r+1} > T,\ B_r + (T - T_r)(c-r) \leq b \right)$$

letting $J_r = B_r + (T - T_r)(c-r)$, equation (17) can be written as

$$Prob\left(T_r \leq T,\ T_{r+1} > T,\ J_r \leq b \right)$$

and this probability is equal to:

$$\int_0^T dx_1 \int_T^\infty dx_2 \int_0^b f(x_1, x_2, x_3)\, dx_3$$

where $f(x_1, x_2, x_3)$ is the joint density of T_r, T_{r+1}, J_r.

Let the Laplace Transform of this density be:

$$f^*(u_1, u_2, u_3) = e^{-u_3(c-r)T} \, \mathbb{E}\left\{ e^{-u_1 T_r - u_2 T_{r+1} - u_3 [J_r - (c-r)T]} \right\}, \qquad (18)$$

and noting the following relations:

$$T_r = \sum_{j=1}^{r} S_j$$

$$T_{r+1} = \sum_{j=1}^{r+1} S_j$$

$$J_r - (c-r)T = \sum_{j=1}^{r} (r-j+1) S_j$$

and hence make the following transformation of variables in equation (1):

$$s_j = u_1 + u_2 + u_3(r-j+1) \quad , \; j = 1, 2, \cdots, r$$

$$s_{r+1} = u_2$$

gives the Laplace Transform of $f(x_1, x_2, x_3)$ as:

$$f^*(u_1, u_2, u_3) = \lambda^{r+1} (c-r) \frac{c!}{(c-r)!} \frac{e^{-u_3(c-r)T}}{(u_3+\lambda)^{r-1}} \cdot$$

$$\frac{1}{u_2 + (c-r)\lambda} \sum_{j=0}^{r-1} \frac{(-1)^{r-j-1}}{(r-j-1)! \, j! \, (u_1 + u_2 + u_3(r-j) + (c-j)\lambda)}$$

Now inverting on u_1 and denoting the result $I_{u_1} f^*$, yields:

$$I_{u_1} f^* = \lambda^{r+1} (c-r) r \binom{c}{r} \frac{e^{-u_3 (c-r) T}}{(u_3+\lambda)^{r-1} \quad (u_2 + (c-r)\lambda)} \cdot$$

$$\sum_{j=0}^{r-1} (-1)^{r-j-1} \binom{r-1}{j} e^{-x_1 [u_2 + u_3 (r-j) + (c-j)\lambda]}$$

Now by factoring $e^{-x_1 [u_2 + u_3 r + c\lambda]}$ from under the summation sign, and applying the binomial theorem to what is left, yields:

$$Iu_1 f^* = \lambda^{r+1} (c-r) r \binom{c}{r} \frac{e^{-u_3 (c-r) T} \quad e^{-x_1 u_3 r} \quad e^{-x_1 u_2}}{(u_3+\lambda)^{r-1} \quad [u_2 + (c-r)\lambda]} \cdot$$

$$[e^{x_1 (u_3+\lambda)} - 1]^{r-1}$$

Next invert on u_2:

$Iu_1 u_2 f^* =$

$$\lambda^{r+1} (c-r) r \binom{c}{r} \frac{e^{-u_3 (c-r) T}}{(u_3+\lambda)^{r-1}} e^{-x_1 u_3 r} \; e^{-x_2 (c-r)\lambda} \cdot$$

$$e^{-x_1 r \lambda} [e^{x_1 (u_3+\lambda)} - 1]^{r-1} U(x_2 - x_1)$$

$$= \lambda^{r+1} (c-r) r \binom{c}{r} \frac{e^{-u_3 (c-r) T}}{(u_3+\lambda)^{r-1}} e^{-x_2 (c-r)\lambda} \cdot$$

$$e^{-x_1 (u_3+\lambda)} [1 - e^{-x_1 (u_3+\lambda)}]^{r-1} U(x_2 - x_1)$$

Hence

$$\int_0^T \int_T^\infty I_{u_1 u_2} f^* \, dx_2 \, dx_1 =$$

$$\lambda^r \binom{c}{r} \frac{e^{-u_3(c-r)T}}{(u_3+\lambda)^r} e^{-T(c-r)\lambda} \left[1 - e^{-T(u_3+\lambda)}\right]^r \tag{19}$$

which can be rewritten as:

$$\lambda^r \binom{c}{r} \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} \frac{e^{-T(c-j)(u_3+\lambda)}}{(u_3+\lambda)^r} \tag{20}$$

Therefore, upon performing the final inversion:

$$f_{N_T^{c},\, B_T}(r,b) = e^{-\lambda b} \lambda^r \binom{c}{r} \cdot$$

$$\sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} \frac{[b - T(c-j)]^{r-1}}{(r-1)!} U(b - T(c-j)), \tag{21}$$

$$b \leq cT$$

which is the desired joint density.

Now by considering the fact that

$$\text{Prob}\{N_T = r\} = \binom{c}{r} e^{-\lambda T(c-r)} \left[1 - e^{-\lambda T}\right]^r$$

and dividing this probability into equation (19), yields:

$$E\left\{e^{-u_3 B_T} \middle| N_T = r\right\} = \left[\frac{1 - e^{-T(u_3+\lambda)}}{u_3 + \lambda}\right]^r \lambda^r e^{-u_3(c-r)T}$$

and noting

$$E\left\{e^{-s N_T - u_3 B_T}\right\} = \sum_{r=0}^{c} e^{-sr} \operatorname{Prob}\left\{N_T = r\right\} E\left\{e^{-u_3 B_T} \,\middle|\, N_T = r\right\}$$

$$= \sum_{r=0}^{c} \binom{c}{r} \lambda^r e^{-sr} e^{-\lambda c T} \frac{e^{-u_3 c T}}{(u_3+\lambda)^r} \left[e^{T(u_3+\lambda)} - 1\right]^r, \qquad (22)$$

which can be rewritten as:

$$e^{-cT(u_3+\lambda)} \left\{ 1 + \frac{\lambda e^{-s}\left[e^{T(u_3+\lambda)} - 1\right]}{(u_3+\lambda)} \right\}^c, \qquad (23)$$

The first and second moments are obtained from equation (22) as:

$$E\{N_T\} = c\left[1 - e^{-\lambda T}\right]$$

$$E\{B_T\} = cT - \frac{c}{\lambda}\left\{\lambda T - 1 + e^{-\lambda T}\right\} \qquad (24)$$

$$\operatorname{Var}\{N_T\} = c e^{-\lambda T}\left\{1 - e^{-\lambda T}\right\}$$

$$\operatorname{Var}\{B_T\} = \frac{c}{\lambda^2}\left\{1 - 2\lambda T e^{-\lambda T} - e^{-2\lambda T}\right\}$$

$$\operatorname{Cov}\{N_T, B_T\} = \frac{c}{\lambda} e^{-\lambda T}\left\{1 - e^{-\lambda T} - \lambda T\right\}$$

For purposes of marking progress when the elapsed time of the testing is $\tau < T$, the memoryless property of the exponential distribution again makes the updating of probability statements relatively easy. Given $N_{\tau} = m$, $B_{\tau} = b_0$, the distribution of N_T and B_T can be obtained by replacing r by r−m, c by c−m, T with $T - \tau$ and translating $N_{T-\tau} = r-m$, $B_{T-\tau} = x - b_0$.

At this point the reliability program manager is again armed with all the instrumentation required to have continuing estimates of the most important characteristics of the test program. It should be noted for this case however, that the estimator for λ, say $\hat{\lambda}$, is now the ratio of two random variables and hence it is this ratio which must be less than the λ_0 designated in the contractual specifications, i.e.:

$$\frac{N_T}{B_T} \leq \lambda_0$$

III. THE NUMBER OF ITEMS TO BE TESTED EXCEEDS THE NUMBER OF TEST CHAMBERS AVAILABLE

A. STOPPING RULE: STOP AT r FAILURES

The development in this chapter will generally follow the development of Chapter II. Testing takes place with replacement as long as possible and testing is complete at the r^{th} failure. At the start of the testing, there are i items available for testing but only c test chambers available, where $i > c$. This chapter is restricted to the case where $r > p = i-c$. To start, let:

T_1 = Time to first failure from $t = 0$

T_2 = Time to second failure from $t = 0$

.

.

T_r = Time to r^{th} failure from $t = 0$

and $S_1 = T_1$

$S_2 = T_2 - T_1$

.

.

$S_r = T_r - T_{r-1}$

then S_j has density $c\lambda e^{-c\lambda x}$, $x > 0$ for $j = 1, 2, \ldots, p$ and has density $(c+p-j+1)\lambda e^{-(c+p-j+1)\lambda x}$, $x > 0$ for $j = p+1, \ldots, r$.

Hence the Laplace Transform of the joint density of the S_j's is:

$$f^*(s_1, s_2, \ldots, s_r) = \prod_{j=1}^{p} \frac{c\lambda}{s_j + c\lambda} \prod_{j=p+1}^{r} \frac{(c+p-j+1)\lambda}{s_j + (c+p-j+1)\lambda} \tag{25}$$

Therefore the Laplace Transform of the joint density of $T_1, T_2, \ldots, T_r$ becomes:

$$f^*(u_1, u_2, \ldots, u_r) = \prod_{j=1}^{p} \frac{c\lambda}{c\lambda + \sum_{k=j}^{r} u_k} \prod_{j=p+1}^{r} \frac{(c+p-j+1)\lambda}{(c+p-j+1)\lambda + \sum_{k=j}^{r} u_k} \tag{26}$$

Now inverting equation (26) on u_1 yields:

$$(c\lambda)^p \frac{e^{-(c\lambda + u_2 + \sum_{l=3}^{r} u_l) t_1}}{(c\lambda + u_2 + \sum_{l=3}^{r} u_l)} \prod_{j=3}^{p} \frac{1}{c\lambda + \sum_{k=j}^{r} u_k} \cdot$$

$$\prod_{j=p+1}^{r} \frac{(c+p-j+1)\lambda}{\lambda(c+p-j+1) + \sum_{k=j}^{r} u_k}$$

Hence the inversion on $u_1, u_2, \ldots, u_p$ produces:

$$f^*(t_1, t_2, \ldots t_p, u_{p+1}, \ldots u_r) =$$

$$(c\lambda)^p e^{-(c\lambda + \sum_{l=p+1}^{r} u_l) t_p} \prod_{j=p+1}^{r} \frac{(c+p-j+1)\lambda}{(c+p-j+1)\lambda + \sum_{k=j}^{r} u_k}$$

and $0 < t_1 < t_2 \quad \ldots \quad < t_p$

and the format has now become that of equation (2) and hence the joint density of $T_1, T_2, \ldots, T_r$ is

$$f(t_1, t_2, \ldots, t_r) = (c\lambda)^p \frac{c!\, \lambda^{r-p}}{(c-r+p)!} e^{-\lambda \sum_{j=p+1}^{r-1} t_j} e^{-(c-r+p+1)\lambda t_r}, \tag{27}$$

The maximum likelihood estimator for λ Using a stopping rule based on r failures and assuming $r > p$ is developed in Appendix B as:

$$\hat{\lambda} = \frac{r}{B_r}$$

Thus again the joint distribution of $\hat{\lambda}$ and the time to completion is tantamount to the joint distribution of T_r, B_r.

The development of this chapter differs slightly from that of Chapter II in that the Laplace Transform of the joint density of T_r, T_m, T_{m+1}, B_r, and J_m (to be defined later) is developed and from this Laplace Transform the desired combinations of distributions are extracted. This method allows for a clear presentation of the relations existing between the various random variables involved in the process.

Consider the joint distribution of T_r, B_r, N_τ, B_τ and again note:

$$\text{Prob}\left(T_r \leq t,\ B_r \leq b,\ B_\tau \leq z,\ N_\tau = m\right) =$$

$$\text{Prob}\left(T_r \leq t,\ T_m \leq \tau,\ T_{m+1} > \tau,\ B_r \leq b,\ J_m \leq z\right)$$

where $J_m = B_m + (\tau - T_m)(i-m)$ for $p < m < i$.

(The case $m \leq p$ is straightforward and will not be treated explicitely.)

This probability is equal to:

$$\int_0^\tau dx_1 \int_0^\tau dx_2 \int_\tau^\infty dx_3 \int_0^b dx_4 \int_0^z dx_5\ f(x_1, x_2, x_3, x_4, x_5)$$

where $f(x_1, \ldots, x_5)$ is the joint density of T_r, T_m, T_{m+1}, B_r, J_m.

Now let the Laplace Transform of this density be:

$f^*(u_1, \ldots, u_5) =$

$$e^{-u_5(i-m)\tau} \, E\left\{ e^{-u_1 T_r - u_2 T_m - u_3 T_{m+1} - u_4 B_r - u_5 (J_m - (i-m)\tau)} \right\} \qquad (28)$$

and note the following relations:

$$T_r = \sum_{j=1}^{r} S_j$$

$$T_m = \sum_{j=1}^{m} S_j$$

$$T_{m+1} = \sum_{j=1}^{m+1} S_j$$

$$B_r = c \sum_{j=1}^{p} s_j + \sum_{j=p+1}^{r} (c+p-j+1) S_j$$

$$J_m - (i-m)\tau = (m-p \sum_{j=1}^{p} S_j + \sum_{j=p+1}^{m} (m-j+1) S_j$$

Hence the transformation of variables

$s_j = u_1 + u_2 + u_3 + cu_4 + (m-p)u_5$ $\qquad j=1, \ldots, p$

$s_j = u_1 + u_2 + u_3 + (c-p-j+1)u_4 + (m-j+1)u_5$ $\qquad j=p+1, \ldots, m$

$s_j = u_1 + u_3 + (c+p-m)u_4$ $\qquad j=m+1$

$s_j = u_1 + (c+p-j+1)u_4$ $\qquad j=m+2, \ldots, r$

can be applied to equation (28) and (noting that $c+p = i$), the Laplace Transform of equation (28) becomes:

$f^*(u_1, \ldots, u_5) =$

$$e^{-u_5(i-m)\tau} \left(\frac{c\lambda}{u_1 + u_2 + u_3 + (u_4+\lambda)c + (m-p)u_5} \right)^{p+1} \cdot$$

$$\prod_{j=p+2}^{m} \frac{(i-j+1)\lambda}{u_1 + u_2 + u_3 + (u_4+\lambda)(i-j+1) + u_5(m-j+1)} \cdot$$

$$\frac{(i-m)\lambda}{u_1 + u_3 + (u_4+\lambda)(i-m)} \cdot \tag{29}$$

$$\prod_{j=m+2}^{r} \frac{(i-j+1)\lambda}{u_1 + (u_4+\lambda)(i-j+1)}$$

Now setting $u_2 = u_3 = u_4 = u_5 = 0$ yields the Laplace Transform of density of T_r to be:

$$f^*(u_1, 0, 0, 0, 0) = \left(\frac{c\lambda}{u_1 + c\lambda} \right)^{p+1} \prod_{j=p+2}^{r} \frac{(i-j+1)\lambda}{u_1 + (i-j+1)\lambda} \tag{30}$$

Equation (30) shows that T_r has an Erlang distribution with r stages and the indicated rates. Now equation (30) can be written as:

$f^*(u_1, 0, 0, 0, 0) =$

$$(c\lambda) \frac{(i-p-1)!}{(i-r)!} \lambda \sum_{j=p+2}^{r} z_j \frac{1}{(u_1+c\lambda)^{p+1} (u_1 + (i-j+1)\lambda)} \tag{31}$$

Where $$z_j = \frac{(-1)^{r-j}}{(j-p-2)!\,(r-j)!}$$

and using theorem 1 of Appendix A, the density of T_r becomes:

$$f(t) = c^{p+1} \frac{(i-p-1)!}{(i-r)!} \lambda^{p+2} \sum_{j=p+2}^{r} z_j \left(\frac{e^{-(i-j+1)\lambda t}}{(j-p-1)^{p+1}} + \right. \tag{32}$$

$$\left. \frac{e^{-c\lambda t}}{(p-j+1)^{p+1}} \sum_{k=0}^{p} (-1)^k \frac{(p-j+1)^{p-k} \lambda^{p-k} t^{p-k}}{(p-k)!} \right)$$

Similarly, the setting of $u_1 = u_2 = u_3 = u_5 = 0$ in equation (29) yields the Laplace Transform of the density of B_r to be:

$$f^*(0,0,0,u_4,0) = \left(\frac{\lambda}{u_4 + \lambda} \right)^r \tag{33}$$

which is the Laplace Transform of a special Erlang distribution with r stages and rate λ and hence the density of B_r is:

$$f(x) = \frac{\lambda (\lambda x)^{r-1} e^{-\lambda x}}{(r-1)!} \quad , \quad x > 0 \tag{34}$$

Now considering equation (29) with $u_2 = u_3 = u_5 = 0$, the Laplace Transform of the joint density of T_r, B_r is produced:

$f^*(u_1, 0, 0, u_4, 0) =$

$$\left(\frac{c\lambda}{u_1 + (u_4 + \lambda) c} \right)^{p+1} \prod_{j=p+2}^{r} \frac{(i-j+1)\lambda}{u_1 + (i-j+1)(u_4 + \lambda)} \tag{35}$$

which can be rewritten as:

$$\lambda^r \, c^{p+1} \frac{(i-p-1)!}{(i-r)!} \quad \frac{1}{[u_1 + (u_4+\lambda)c]^{p+1}} \quad \frac{1}{(u_4+\lambda)^{r-p-2}} \cdot$$

$$\sum_{j=p+1}^{r-1} \frac{(-1)^{r-j-1}}{(j-p-1)!\,(r-j-1)!} \cdot \frac{1}{[u_1 + (u_4+\lambda)(i-j-1)]}$$

therefore:

$Iu_1 \, f^* =$ (36)

$$\lambda^r \, c^{p+1} \frac{(i-p-1)!}{(i-r)!} \quad \frac{1}{(u_4+\lambda)^{r-p-2}} \quad \sum_{j=p+1}^{r-1} \frac{(-1)^{r-j-1}}{(j-p-1)!\,(r-j-1)!} \cdot$$

$$\frac{e^{-x_1[(u_4+\lambda)(i-j)]}}{(u_4+\lambda)^{p+1}\,(j-p)^{p+1}} \quad +$$

$$\frac{e^{-x_1(u_4+\lambda)c}}{(u_4+\lambda)^{p+1}\,(p-j)^{p+1}} \quad \sum_{k=0}^{p} (-1)^k \left\{ x_1 (u_4+\lambda)(p-j) \right\}^{p-k} \frac{1}{(p-k)!}$$

Now by inverting on u_4, the joint density of $T_r\ B_r$ is obtained as:

$$I_{u_1\, u_4}\ f^* = f_{T_r,\, B_r}(x_1\ x_4) = \tag{37}$$

$$\lambda^r c^{p+1} \frac{(i-p-1)!}{(i-r)!} \sum_{j=p+1}^{r-1} (-1)^{r-j-1} \frac{1}{(j-p-1)!\,(r-j)!} \Bigg\{$$

$$e^{-x_4\lambda} \frac{(x_4 - x_1(i-j))^{r-2}\ U(x_4 - x_1(i-j))}{(r-2)!\ (j-p)^{p+1}} +$$

$$\sum_{k=0}^{p} (-1)^k e^{-x_4\lambda} \frac{(x_4 - x_1 c)^{r-p+k-2}\ U(x_4 - x_1 c)}{(r-p+k-2)!\ (p-k)!\ (p-j)^{k+1}} \Bigg\}$$

By considering the logarithm of the transform, Read [Ref.5] has shown:

$$E(T_r) = \frac{p}{c\lambda} + \frac{1}{\lambda} \sum_{j=0}^{r-p-1} \frac{1}{(c-j)}$$

$$VAR(T_r) = \frac{p}{(c\lambda)^2} + \frac{1}{\lambda^2} \sum_{j=0}^{r-p-1} \frac{1}{(c-j)^2} \tag{38}$$

$$E(B_r) = \frac{r}{\lambda}$$

$$VAR(B_r) = \frac{r}{\lambda^2}$$

$$Cov(T_r, B_r) = \frac{pc}{\lambda^2} + \frac{1}{\lambda^2} \sum_{j}^{r-p-1} \frac{1}{(c-j)}$$

The program manager is now armed with the required distribution theory to determine if the reliability testing has successfully demonstrated the specified contractual reliability. If the manager desires to mark progress prior to the completion of the testing, the data available at some time, τ , will be a complete history of the testing including N_τ , B_τ . Again the memoryless property of the exponential distribution makes N_τ , B_τ the only relevant data in terms of making probability statements concerning the density of T_r, B_r. Appendix B develops the maximum likelihood estimator for λ , say λ_τ , for a fixed time τ , as:

$$\lambda_\tau = \frac{N_\tau}{B_\tau}$$

It also follows that the conditional density of T_r, B_r given $N_\tau = m$ (for $m < r$) and $B_\tau = b$ can be characterized from equations (32) and (34) by replacing r with (r−m), i with (i−m) and by translating T_r to $T_{r-m} + \tau$, and B_r to $B_{r-m} + b$, keeping track of whether $m \leq p$ or $m > p$. Thus:

$$f^*_{T_r, B_r \mid N_\tau = m, B_\tau = b}(u_1, u_4) =$$

$$e^{-u_1 \tau - u_4 b} \, f^*(u_1, 0, 0, u_4, 0)$$

with the appropriate translations on the indices depending on the value of m.

The conditional moments are given by:

$$E\left\{T_r \mid N_\tau = m\ (<p)\right\} = \tau + \frac{p-m}{c\lambda} + \frac{1}{\lambda}\sum_{j=0}^{r-p-1}\frac{1}{(c-j)}$$

$$E\left\{T_r \mid N_\tau = m\ (>p)\right\} = \tau + \frac{1}{\lambda}\sum_{j=m-p}^{r-p-1}\frac{1}{(c-j)}$$

$$VAR\left\{T_r \mid N_\tau = m\ (<p)\right\} = \frac{p-m}{(c\lambda)^2} + \frac{1}{\lambda^2}\sum_{j=0}^{r-p-1}\frac{1}{(c-j)^2} \tag{39}$$

$$VAR\left\{T_r \mid N_\tau = m\ (>p)\right\} = \frac{1}{\lambda^2}\sum_{j=m-p}^{r-p-1}\frac{1}{(c-j)^2}$$

$$E\left\{B_r \mid N_\tau = m,\ B_\tau = b\right\} = b + \frac{r-m}{\lambda}$$

$$VAR\left\{B_r \mid N_\tau = m,\ B_\tau = b\right\} = \frac{r-m}{\lambda^2}$$

Again the manager is confronted with a maximal failure rate as derived from the contractual specifications on reliability. The system or equipment under test in this case must develop a total busy time B_r which is greater than r/λ_o . At this point the manager is equipped with the necessary managerial aids for decision making throughout the test program.

B. STOPPING RULE: STOP AT A FIXED TIME T

Under a fixed time stopping rule, with $i > c$, the maximum likelihood estimator for λ upon completion of the test program is:

$$\hat{\lambda} = \frac{N_T}{B_T}$$

and again the joint distribution of N_T, B_T is required for experimental design and decision making. Using the same convention as in Chapter II yields the joint density on N_T, B_T as:

$$f_{N_T, B_T}(m,b) = \frac{\partial \, Prob\,(N_T = m\,,\, B_T \leq b)}{\partial b}$$

in this case however the procedure is not quite so straightforward as it was in Chapter II. Equation (29) requires some understanding of the indices. Proper manipulation of the indices yields:

$$E\left\{ e^{-U_2 T_m - U_3 T_{m+1} - U_5 J_m} \right\} =$$

$$e^{-U_5 c T} \left[\frac{c\lambda}{U_2 + U_3 + c\lambda} \right]^{m+1} \frac{c\lambda}{U_3 + c\lambda}$$

where $m \leq p$.

It should be noted that B_T is a degenerate random variable for this case.

$$E\left\{ e^{-U_2 T_m - U_3 T_{m+1} - U_5 J_m} \right\} =$$

$$e^{-U_5 (c-1) T} \left[\frac{c\lambda}{U_2 + U_3 + U_5 + c\lambda} \right]^{p+1} \frac{(c-1)\lambda}{U_3 + (c-1)\lambda}$$

Where $m = p + 1$.

$$E\left\{e^{-u_2 T_m - u_3 T_{m+1} - u_5 J_m}\right\} = \tag{40}$$

$$e^{-u_5(i-m)T} \left[\frac{c\lambda}{u_2 + u_3 + (m-p)u_5 + c\lambda}\right]^{p+1} \cdot$$

$$\prod_{j=p+2}^{m} \frac{(i-j+1)\lambda}{u_2 + u_3 + (m-j+1)u_5 + (i-j+1)\lambda} \cdot$$

$$\frac{(i-m)\lambda}{u_3 + (i-m)\lambda}$$

where $p+1 < m < i$

The following relationships should be noted

$$E\left\{e^{-u_5 B_T} \mid N_T = m\right\} = \text{Equation (40) with } u_2\ u_3 = 0$$

$$E\left\{e^{-u_5 B_T}\right\} = \sum_{r=0}^{\infty} \text{Prob}\left\{N_T = r\right\} E\left\{e^{-u_5 B_T} \mid N_T = r\right\},$$

$$E\left\{e^{-u N_T - u_5 B_T}\right\} = \sum_{r=0}^{\infty} \text{Prob}\left\{N_T = r\right\} e^{-ur} E\left\{e^{-u_5 B_T} \mid N_T = r\right\}$$

The marginal distribution of N_T is obtainable by considering for each r, r = 0, 1, 2, ... i , the probability N_T = r. For r = 0, 1, 2, ... p, the process is a Poisson process with rate .

Hence:

$$\text{Prob}\left\{N_T = r\right\} = \frac{(c\lambda T)^r e^{-c\lambda T}}{r!} \tag{41}$$

$$r = 0, 1, \ldots p$$

For r = p + 1

and from equation (40), the Laplace Transform of this joint density is:

$$f^*(u_2, u_3) = \frac{c\lambda}{u_2 + u_3 + c\lambda} \qquad \frac{(c-1)\lambda}{u_3 + (c-1)\lambda}$$

Inversion in u_2 yields:

$$I_{u_2} f^* = \frac{c^{p+1} (c-1) \lambda^{p+2} \; X_2^{\,p} \quad e^{-X_2 (u_3 + c\lambda)}}{p! \qquad [u_3 + c\lambda]}$$

Inversion on u_3 yields:

$$I_{u_2 u_3} f^* = \frac{c^{p+1} (c-1) \lambda^{p+2} \; X_2^{\,p} \; e^{-X_2 \lambda} \quad e^{-X_3 (c-1)\lambda}}{p!}$$

$$X_3 > X_2$$

Now:

$$\int_T^\infty dx_3 \; I \, u_2 u_3 \, f^* =$$

$$\frac{c^{p+1} \lambda^{p+1} x_2^p e^{-x_2 \lambda} e^{-T(c-1)\lambda}}{p!}$$

$$0 < x_2 < T$$

Finally,

$$P\{N_T = p+1\} = c^{p+1} e^{-T(c-1)\lambda} \left[1 - e^{-\lambda T} \sum_{\ell=0}^{p} \frac{(\lambda T)^{p-\ell}}{(p-\ell)!} \right], \tag{42}$$

Now for r $p + 1$, the Laplace Transform of the joint density of T_r, T_{r+1} is given by equation (40) as:

$$f^*(u_2, u_3) = \left[\frac{c\lambda}{u_2 + u_3 + c\lambda} \right]^{p+1} \cdot \prod_{j=p+2}^{r} \frac{(i-j+1)\lambda}{u_2 + u_3 + (i-j+1)\lambda} \cdot \frac{(i-r)\lambda}{u_3 + (i-r)\lambda} \tag{43}$$

Rewriting gives:

$$c^{\rho+1}\lambda^{r+1}\frac{(i-\rho-1)!}{(i-r-1)!}\ \frac{1}{u_3+(i-r)\lambda}\sum_{j=\rho+2}^{r}(-1)^{r-j}\ \frac{1}{(j-\rho-2)!\,(r-j)!}\cdot$$

$$\left\{\frac{1}{(u_2+u_3+c\lambda)^{\rho+1}}\cdot\frac{1}{u_2+u_3+(i-j+1)\lambda}\right\}$$

$$I_{u_2}f^* = c^{\rho+1}\lambda^{r+1}\frac{(i-\rho-1)!}{(i-r-1)!}\ \frac{1}{u_3+(i-r)\lambda}\sum_{j=\rho+2}^{r}(-1)^{r-j}\cdot$$

$$\frac{1}{(j-\rho-2)!\,(r-j)!}\left\{\frac{e^{-x_2[u_3+(i-j+1)\lambda]}}{(\rho+1-j)^{\rho+1}\lambda^{\rho+1}}+\right.$$

$$\left.\frac{e^{-x_2[u_3+c\lambda]}}{(\rho+1-j)^{\rho+1}\lambda^{\rho+1}}\sum_{k=0}^{\rho}(-1)^k\ \frac{[x_2(\rho+1-j)\lambda]^{\rho-k}}{(\rho-k)!}\right\}$$

The only terms involving u_3 are

$$\frac{e^{-x_2u_3}}{u_3+(i-r)\lambda}$$

and the inversion of this term yields:

$$e^{x_2(i-r)\lambda}\quad e^{-x_3(i-r)\lambda}\quad U(x_3-x_2)$$

The only term involving x_3 is in the previous inversion, hence

$$\int_T^{\infty} I_{u_2 u_3} f^* \, dx_3 =$$

$$e^{-T(i-r)\lambda} e^{p+1} \lambda^r \frac{(i-p-1)!}{(i-r)!} \sum_{j=p+2}^{r} \frac{(-1)^{r-j}}{(j-p-2)!\,(r-j)!} \Bigg\{$$

$$\frac{e^{-x_2(r-j+1)\lambda}}{(j-p-1)^{p+1}\lambda^{p+1}} \quad +$$

$$\frac{e^{-x_2(r-p)\lambda}}{(p+1-j)^{p+1}\lambda^{p+1}} \sum_{k=0}^{p} \frac{(-1)^k \left[x_2(p+1-j)\lambda\right]^{p-k}}{(p-k)!} \Bigg\}$$

The first term involving x_2 is

$$e^{-x_2(r-j+1)\lambda}$$

and

$$\int_0^T e^{-x_2(r-j+1)\lambda} \, dx_2 = \frac{1-e^{-T(r-j+1)\lambda}}{(r-j+1)\lambda}.$$

The next term is to note for K = p

$$\int_0^T (-1)^p \frac{e^{-x_2(r-p)\lambda}}{(p+1-j)^{p+1}\lambda^{p+1}} = (-1)^p \frac{\left[1-e^{-T(r-p)\lambda}\right]}{(p+1-j)^{p+1}\lambda^{p+2}(r-p)}$$

and for all other terms inside the sum

$$\sum_{k=0}^{p-1} \int_0^T \frac{(-1)^k}{(p+1)^{k+1}\lambda^{k+1}} \cdot \frac{x_2^{p-k}\, e^{-x_2(r-p)\lambda}}{(p-k)!} =$$

$$\sum_{k=0}^{p-1} \Bigg\{ (-1)^k \frac{1}{(p+1-j)^{k+1}\lambda^{p+2}(r-p)^{p-k+1}} +$$

$$(-1)^{k+1} \frac{e^{-T(r-p)\lambda}}{(p+1-j)^{k+1}\lambda^{k+1}} \cdot$$

$$\sum_{1=0}^{p-k} \frac{T^{p-k-1}}{(p-k-1)!\,[(r-p)\lambda]^{1+1}} \Bigg\}$$

Therefore the probability that $N_T = r$ is:

$$e^{-T(i-r)\lambda} \frac{c^p c!\, \lambda^r}{(i-r)!} \sum_{j=p+2}^{r} (-1)^{r-j} \frac{1}{(j-p-2)!\,(r-j)!} \Bigg\{$$

$$\frac{1-e^{-T(r-j+1)\lambda}}{(j-p-1)^{p+1} \lambda^{p+2} (r-j+1)} + \frac{(-1)^p \left[1-e^{-T(r-p)\lambda}\right]}{(p+1-j)^{p+1} \lambda^{p+2} (r-p)} + \tag{44}$$

$$\sum_{k=0}^{p-1} \left\{ \frac{(-1)^k}{(r-p)^{p-k+1} (p+1-j)^{k+1} \lambda^{p+2}} + \frac{(-1)^{k+1} e^{-T(r-p)\lambda}}{(p+1-j)^{k+1} \lambda^{k+1}} \sum_{l=0}^{p-k} \frac{T^{p-k-l}}{(p-k-1)!\,[(r-p)\lambda]^{l+1}} \right\} \Bigg\}$$

From the equations thus far derives the moments of the bivariate distribution can be developed as

$$E\{N_T\} = \sum_{r=0}^{i} r\, P\{N_T = r\}$$

$$E\{B_T\} = -\left.\frac{\partial E\left(e^{-u_5 B_T}\right)}{\partial u_5}\right|_{u_5=0} \tag{45}$$

$$E\{N_T^2\} = \sum_{r=0}^{i} r^2\, P(N_T = r)$$

$$E\{B^2\} = \left.\frac{\partial^2 E\left(e^{-u_5 B_T}\right)}{\partial u_5^2}\right|_{u_5=0}$$

Now, in view of the convention for $f_{N_T, B_T}(m,b)$, this bivariate density can be derived from equation (40), in that

$$f_{N_T, B_T}(m,b) = \int_0^T dx_2 \int_T^\infty dx_3\, I_{u_2 u_3 u_5}\, f^* \tag{46}$$

Where f^* is the appropriate Laplace Transform depending on the value of m. To start let $m \leq p$ and hence

$$f^*(u_2, u_3, u_5) = e^{-u_5 cT} \left[\frac{c\lambda}{u_2 + u_3 + c\lambda} \right]^{m+1} \frac{c\lambda}{u_3 + c\lambda}$$

Where $e^{-u_5 cT}$ is the Laplace Transform of the degenerate random variable B_T. Therefore:

$$f_{N_T, B_T}(m, b) = \frac{cT\,(c\lambda T)^m e^{-c\lambda T}}{m!} \tag{47}$$

When m = 0, 1, 2, ... p; and b=cT

Now for m = p+1

$$f^*(u_2, u_3, u_5) = e^{-u_5 (c-1)T} \left[\frac{c\lambda}{u_2 + u_3 + u_5 + c\lambda} \right]^{p+1} \frac{(c-1)\lambda}{u_3 + (c-1)\lambda}$$

$$I_{u_2} f^* = \frac{e^{-u_5 (c-1)T} c^{p+1} (c-1)\, \lambda^{p+2}\, X_2^p\, e^{-X_2 [u_3 + u_5 + c\lambda]}}{p!}$$

$$I_{u_2 u_3} f^* = \frac{e^{-u_5 (c-1)T} c^{p+1} (c-1)\, \lambda^{p+2}\, X_2^p\, e^{-X_2 (u_5 + \lambda)}\, e^{-X_3 (c-1)\lambda}}{p!}$$

$$X_3 > X_2$$

$$\int_T^{\infty} I_{u_2 u_3} f^* \, dx_3 = e^{-u_5 (c-1)T} (c\lambda)^{p+1} \, x_2^p \, e^{-x_2 (u_5+\lambda)} \, e^{-T(c-1)\lambda}$$

$$\int_0^T dx_2 \int_T^{\infty} dx_3 \, I_{u_2 u_3} f^* = e^{-T(c-1)(u_5+\lambda)} (c\lambda)^{p+1} \cdot$$

$$\left\{ \frac{1}{(u_5+\lambda)^{p+1}} - e^{-T(u_5+\lambda)} \sum_{\ell=0}^{p} \frac{T^{p-\ell}}{(p-\ell)!\,(u_5+\lambda)^{\ell+1}} \right\}$$

Inverting on u_5 yields:

$f_{N_T, B_T}(p+1, b) =$

$$\frac{e^{-b\lambda} (c\lambda)^{p+1} \left[b - T(c-1) \right]^p \, U\left[b - T(c-1) \right]}{p!} \tag{48}$$

It should be noted that another term would have appeared in this inversion except for limitation $(c-1)T \leq b \leq cT$.

By a development similar to that of equation (48), it can be shown:

$$f_{N_T, B_T}(m, b) =$$

$$\frac{e^{-b\lambda} c^p \lambda^m c!}{(i-m)!} \sum_{j=p+2}^{m} \frac{(-1)^{m-j}}{(j-p-2)!\,(m-j)!} \Bigg\{$$

$$\frac{[b - T(i-m)]^{m-1} - [b - T(i-j+1)]^{m-1}}{(j-p-1)^{p+1}\,(m-j-1)\,(m-1)!} + \tag{49}$$

$$\frac{[b - T(i-m)]^{m-1}}{(m-1)!} \left\{ \sum_{k=0}^{p} \frac{(-1)^k}{(p+1-j)^{k+1}\,(m-p)^{p-k+1}} \right\} \Bigg\}$$

Where m = p + 2, p + 3, ..., i and all terms in square brackets are followed by a unit step function with the same argument.

The reliability program manager now has all the distribution theory necessary in this case to assist him in making decisions about the testing program. The formulations look somewhat unmanageable, but with specific numbers these formulas should not be too difficult to solve. If the numbers become large, use of the normal approximation derived by Read [Ref.5] would probably be more practical.

APPENDIX A

THEOREM ONE

Theorem one is stated as follows:

If $f^*(s)$ is the Laplace Transform of $f(t)$ such that

$$f^*(s) = \frac{1}{(s+a)^n (s+b)}, \quad a \neq b, \text{ then}$$

$$f(t) = \frac{e^{-bt}}{(a-b)^n} + \frac{e^{-at}}{(b-a)^n} \sum_{k=0}^{n-1} (-1)^k \frac{[t(b-a)]^{n-1-k}}{(n-1-k)!} \tag{50}$$

Proof:

By the theorem of residues of complex variables,

$$f(t) = \left. \frac{e^{st}}{(s+a)^n} \right|_{s=-b} + \left. \frac{1}{(n-1)!} \left\{ \frac{d^{n-1}\left(\frac{e^{st}}{s+b}\right)}{d\, s^{n-1}} \right\} \right|_{s=-a} \tag{51}$$

Now by the binomial expansion for derivatives of products, if

$$g(s) = \frac{e^{st}}{s+b},$$

then the $(n-1)^{st}$ derivative with respect to s is:

$$g^{n-1}(s) = \frac{e^{st}}{(s+b)^n} \cdot \sum_{j=0}^{n-1} (-1)^j (s+b)^{n-1-j}\, t^{n-1-j} \frac{(n-1)!}{(n-1-j)!} \tag{52}$$

Now substituting equation (52) into equation (51) yields:

$$f(t) = \left. \frac{e^{st}}{(s+a)^{n}} \right|_{s=-b} +$$

$$\left. \frac{1}{(n-1)!} \frac{e^{st}}{(s+b)^{n}} \sum_{j=0}^{n-1} (-1)^{j} (s+b)^{n-1-j} t^{n-1-j} \frac{(n-1)!}{(n-1-j)!} \right|_{s=-a}$$

Hence:

$$f(t) = \frac{e^{-bt}}{(a-b)^{n}} + \frac{e^{-at}}{(b-a)^{n}} \sum_{j=0}^{n-1} (-1)^{j} (b-a)^{n-1-j} t^{n-1-j} \frac{1}{(n-1-j)!}$$

APPENDIX B

MAXIMUM LIKELIHOOD ESTIMATORS FOR THE FAILURE RATE

The estimator $\hat{\lambda}_r$ for λ upon completion of the test program (i.e. upon obtaining r failures) is developed as follows:

The likelihood function, L(λ), based on the samples of S_j, is:

$$L(\lambda) = \prod_{j=1}^{p} c\lambda e^{-c\lambda S_j} \prod_{j=p+1} (c+p-j+1)\lambda e^{-(c+p-j+1)\lambda S_j}$$

$$L(\lambda) = (c\lambda)^p e^{-c\lambda \sum_{j=1}^{p} S_j} \prod_{j=p+1}^{r} (c+p-j+1)\lambda e^{-(c+p-j+1)\lambda S_j}$$

Now letting $\phi(\lambda) = \mathrm{Ln}(L(\lambda))$, yields:

$$\phi(\lambda) = p\,\mathrm{LN}(c\lambda) - c\lambda \sum_{j=1}^{p} S_j + \sum_{j=p+1}^{r} \mathrm{LN}[(c+p-j+1)\lambda]$$

$$- \sum_{j=p+1}^{r} (c+p-j+1)\lambda S_j$$

$$\phi'(\lambda) = \frac{p}{\lambda} - c\sum_{j=1}^{p} S_j + \sum_{j=p+1}^{r} \frac{1}{\lambda} - \sum_{j=p+1}^{r} (c+p-j+1) S_j \,,$$

Setting equation (53) equal to zero and solving for λ yields.

$$\frac{r}{\lambda} = c \sum_{j=1}^{p} S_j + \sum_{j=p+1}^{r} (c+p-j+1) S_j$$

and recognizing the right-hand side as B_r yields the estimator for to be:

$$\hat{\lambda}_r = \frac{r}{B_r} \tag{54}$$

Now the maximum likelihood estimator for λ at some fixed time τ, prior to completion of the testing, where $N_\tau = m < p$ is given by:

$$\hat{\lambda} = \frac{m}{B_\tau}$$

In the case where $N_\tau = m > p$, the following data is available: $T_1, T_2, \ldots, T_m$ and B_τ. Letting:

$$S_1 = T_1$$

$$S_2 = T_2 - T_1$$

$$\cdot$$

$$\cdot$$

$$S_m = T_m - T_{m-1}$$

and noting that the probability of zero failures in $(\tau - T_m)$, as a function of λ, is:

$$e^{-(c-(m-p))\lambda(\tau - T_m)}$$

the likelihood function becomes:

$$L(\lambda) = \prod_{j=1}^{p} c\lambda e^{-c\lambda S_j} \prod_{j=p+1}^{m} (c+p-j+1)\lambda \cdot$$

$$e^{-(c+p-j+1)\lambda S_j - (c-(,--))\lambda(\tau - T_m)}$$

Now letting $\emptyset(\lambda) = Ln(L(\lambda))$, yields:

$$\emptyset(\lambda) = pLn(c\lambda) - c\lambda \sum_{j=1}^{p} S_j + \sum_{j=p+1}^{m} Ln((c+p-j+1)\lambda) -$$

$$\sum_{j=p+1}^{m} (c+p-j+1)\lambda S_j - (c-(m-p))\lambda(\tau - T_m)$$

$$\emptyset'(\lambda) = \frac{p}{\lambda} - c\sum_{j=1}^{p} S_j + \sum_{j=p+1}^{m} \frac{1}{\lambda} - \sum_{j=p+1}^{m} (c+p-j+1)S_j$$

$$- (c-(m-p))(\tau - T_m)$$

Setting $\emptyset'(\lambda) = 0$, yields:

$$\frac{m}{\lambda} = c\sum_{j=1}^{p} S_j = \sum_{j=p+1}^{m} (c+p-j+1)S_j + (c-(m-p))(\tau - T_m)$$

Recognizing the right-hand side as B_τ yields the estimator for λ to be:

$$\hat{\lambda}_\tau = \frac{M}{B_\tau} \tag{55}$$

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13. ABSTRACT

Various aspects of reliability test programs when the number of items to be tested equals or exceeds the number of test chambers available for testing and when the underlying failure distribution is exponential are studied. Specifically, two bivariate processes are formed and characterized; (i) the time until the r^{th} failure and the accumulated test time of the first r failures and (ii) the number of failures in the time interval (O,T) and the total test time accumulated in (O,T). These are treated in the two cases; replacement and nonreplacement. Relations which exist between contractual specifications for reliability demonstration and the underlying distributions of the test program are discussed. Maximum likelihood estimators are developed for the distributions when necessary.

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